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One-Point Extensions of t-Koszul Algebras

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Abstract We introduce the category of t-fold modules which is a full subcategory of graded modules over a graded algebra. We show that this subcategory and hence the subcategory of t-Koszul modules are both closed under extensions and cokernels of monomorphisms. We study the one-point extension algebras, and a necessary and sufficient condition for such an algebra to be t-Koszul is given. We also consider the conditions such that the category of t-Koszul modules and the category of quadratic modules coincide.

Keywords graded algebra, Koszul algebra, Koszul module
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1 Introduction

The t-Koszul algebra was first introduced by Roland Berger in [1] as a generalization of Koszul algebras, where he called it a "generalized" Koszul algebra. It plays an important role in noncommutative geometry. It's well known that 0,1-generated Artin–Schelter regular algebras of global dimension 3 give an important class of t-Koszul algebras. Zhang and the coauthors generalized this kind of algebras to the non-connected case, see [2–4]. Note that in [4], it was called a d-Koszul algebra. In this paper, we follow the definitions and notations given in [2].

In [1–2], the so-called t-Koszul complex for a graded algebra was introduced, and it was shown that a graded algebra is t-Koszul if and only if the associated t-Koszul complex is augmented. This is the fundamental property of t-Koszul algebras. Another important thing is that the Ext-algebra of t-Koszul algebra, after regrading, is also a Koszul algebra, see [4]. In [3], the monomial graded algebras over which the subcategory of t-Koszul modules coincides with the subcategory of modules with linear presentations are classified, they are given by the monomial t-Koszul algebras.

In algebra and ring theory, the extension method is a very important way to construct new algebras or rings from given ones. In [4], the authors consider the Koszulity of one-point extension algebras. They gave a sufficient and necessary condition for such an algebra to be Koszul. These stimulate us to study t-Koszul algebras via the one-point extension method, we

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obtain much more generalized results. We also consider the conditions such that the category of quadratic modules coincides with the category of t-Koszul modules.

The paper is arranged as follows. In Section 2, we give some basic notations and definitions. In Section 3, the category of t-Koszul modules and the category of so-called t-fold modules (see Definitions 2.1 and 2.3) are studied. We show that both categories are closed under direct sum, direct summand, extensions and cokernels of monomorphisms. Note that such categories are not necessarily closed under kernels of epimorphisms, and counter-examples are also given in the paper.

Section 4 deals with the Koszulity of one-point extensions of algebras. We obtain a sufficient and necessary condition for one-point extension algebras to be t-Koszul. It is the main result of this paper, which provides us with an easy way to construct plenty of examples of t-Koszul algebras from the known ones. In Section 5, we consider the conditions such that the category of t-Koszul modules and the category of the quadratic modules coincide. For the one-point extension algebras case, we give a necessary condition and under certain additional conditions a sufficient condition.

2 Notations and Definitions

Let k be any fixed field. A *positively graded k-algebra* is a family $\{\Lambda_i, \phi_{ij}\}_{i,j=0}^{\infty}$ of k-vector spaces and maps satisfying:

(1) Λ_0 is a k-algebra;
(2) Each Λ_i is a finite-dimensional Λ_0-Λ_0-bimodule;
(3) $\phi_{ij} : \Lambda_i \otimes_{\Lambda_0} \Lambda_j \to \Lambda_{i+j}$;
(4) The following diagram commutes:

$$\begin{array}{ccc} \Lambda_i \otimes_{\Lambda_0} \Lambda_j \otimes_{\Lambda_0} \Lambda_k & \xrightarrow{\phi_{ij}\otimes 1} & \Lambda_{i+j} \otimes_{\Lambda_0} \Lambda_k \\ {\scriptstyle 1\otimes\phi_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{(i+j)k}} \\ \Lambda_i \otimes_{\Lambda_0} \Lambda_{j+k} & \xrightarrow{\phi_{i(j+k)}} & \Lambda_{i+j+k}. \end{array}$$

We will write $\{\Lambda_i, \phi_{ij}\}_{i,j=0}^{\infty}$ as $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ and the image of ϕ_{ij} by $\Lambda_i\Lambda_j$.

We assume that Λ is *elementary*, that is, Λ_0 is finite product of copies of k. Furthermore, Λ is always *generated in degree* 0 *and* 1, i.e., $\Lambda_i\Lambda_j = \Lambda_{i+j}$ for all $i, j \geq 0$. Finally, we assume that each Λ_i is a finite-dimensional k-vector space. We call such an algebra an *elementary* $0, 1$-*generated algebra*. All algebras considered here are elementary 0,1-generated.

If Λ is an elementary 0,1-generated algebra, then Λ is isomorphic to kQ/I, where Q is a finite quiver determined by the Λ_0-Λ_0-bimodule structure of Λ_1, kQ is the path algebra of Q over k and I is a two-sided ideal of kQ generated by homogeneous elements with length not less than 2. It follows that Λ is finite-dimensional if and only if I can be chosen to be admissible, i.e, $J^N \subseteq I \subseteq J^2$ for some positive integer $N \geq 2$, where J is the graded Jacobson radical of kQ, which is a two-sided ideal generated by all the arrows of Q.

We call $M = \bigoplus_{i=-\infty}^{\infty} M_i$ a *graded Λ-module* if each M_i is a Λ_0-module and there are linear maps $\Lambda_i \otimes_{\Lambda_0} M_j \to M_{i+j}$ giving M a Λ-module structure. Denote the image of the multiplication map given above by Λ_iM_j. A graded Λ-module M is said to be *generated in degree i* (or M is *i-generated*) provided that $M_j = 0$ for $j < i$ and $M_{i+j} = \Lambda_jM_i$ for all $j \geq 0$. Let M and N be graded Λ-modules. We say that a Λ-homomorphism $f : M \to N$ is of *degree i* if $f(M_j) = M_{i+j}$ for all j. We denote the category of graded Λ-modules with degree 0 maps by $Gr(\Lambda)$ and denote by Λ-Mod the category of all left Λ-modules.

An important subcategory of $Gr(\Lambda)$ is the full subcategory of graded Λ-modules generated in degree 0, which we denote by $Gr_0(\Lambda)$. Note that the projective modules in $Gr_0(\Lambda)$ are the graded summands of a direct sum of copies of Λ.

Henceforth, unless otherwise stated, all modules and graded modules will be finitely generated and we let Λ-mod, $gr(\Lambda)$ and $gr_0(\Lambda)$ denote the subcategories of Λ-Mod, $Gr(\Lambda)$, and

$Gr_0(\Lambda)$, respectively, consisting of finitely generated modules.

Let $M = \bigoplus_{i=-\infty}^{\infty} M_i$ be a graded Λ-module and n be an integer, we let $M[n]$ denote the graded Λ-module $N = \bigoplus_{i=-\infty}^{\infty} N_i$ such that $N_i = M_{i-n}$. Note that Λ can be viewed as a graded Λ-module generated in degree 0. If P is a graded summand of Λ, then $P[n]$ is a graded projective Λ-module. Graded projective Λ-modules are just direct sums of projective modules of the form $P[n]$ for $n \in \mathbf{Z}$.

A graded Λ-module M has a *linear resolution* if there is a graded projective resolution in $gr(\Lambda)$:

$$\cdots \to P_i \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

such that P_i is generated in degree i for $i = 0, 1, 2, \ldots$.

We will denote by $\mathscr{K}(\Lambda)$ the full subcategory of $gr_0(\Lambda)$ consisting of modules with linear resolutions.

A graded algebra $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ is called a *Koszul algebra* if each simple Λ-module has a linear resolution when regarded as a graded module concentrated in degree 0, i.e, if Λ is 0,1-generated, and $\Lambda_0 \cong \Lambda/\mathrm{Rad}(\Lambda)$ has a linear resolution, where $\mathrm{Rad}(\Lambda)$ is the graded Jacobson radical of Λ.

Next we recall the definitions of t-Koszul algebras and t-Koszul modules which play a fundamental role in this paper.

Definition 2.1 *Let Λ be a $0,1$-generated elementary algebra and $t \geq 2$ an integer.*

(i) *A Λ-module M is called a t-Koszul module, if for some integer n, $M[n]$ has a graded projective resolution in $gr_0(\Lambda)$*

$$\cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M[n] \longrightarrow 0,$$

such that P_i is $t(i)$-generated, where

$$t(i) = \begin{cases} tm+1, & i = 2m+1; \\ tm, & i = 2m, \end{cases} \qquad m \in \boldsymbol{Z}.$$

In this case, M is called an n-generated t-Koszul module. Unless otherwise noted, a t-Koszul module always means a 0-generated t-Koszul module;

(ii) *If Λ_0 is a t-Koszul module, then we call Λ a t-Koszul algebra.*

Remark 2.2 Let Q be a finite quiver and $t \geq 2$ an integer.

(1) The path algebra kQ and the algebra kQ/J^t are both t-Koszul algebras, where J is the graded Jacobson radical of kQ which is generated by kQ_1;

(2) If $t = 2$, then we get the usual Koszul algebras and modules.

Definition 2.3 *Let Λ be a $0,1$-generated elementary algebra and $t \geq 2$ an integer. A graded Λ-module M is called a t-fold Λ-module if there is a projective presentation in $gr(\Lambda)$:*

$$P_1 \to P_0 \to M \to 0,$$

where P_0 and P_1 are projective modules generated in degree 0 and degree $t-1$, respectively. Let n be an integer, we call $M \in gr(\Lambda)$ an n-generated t-fold module if $M[n]$ is a t-fold module. We say that a graded algebra Λ is a t-$fold$ algebra if $\mathrm{Rad}(\Lambda)$ is a 1-generated t-fold module.

Note that 2-fold modules and algebras are just the quadratic ones defined in [6].

Denote by $\mathscr{L}_t(\Lambda)$ the full subcategory of $gr(\Lambda)$ consisting of all t-fold Λ-modules. When $t = 2$, we omit the subscript t, i.e., $\mathscr{L}(\Lambda) = \mathscr{L}_2(\Lambda)$.

Remark 2.4 (1) Clearly $\mathscr{K}_t(\Lambda) \subseteq \mathscr{L}(\Lambda) \subseteq gr_0(\Lambda)$ and $\mathscr{L}_t(\Lambda) \subseteq gr_0(\Lambda)$ for any $t \geq 2$.

(2) $M \in \mathscr{K}_t(\Lambda)$ if and only if, for any integer $n \geq 0$, $\Omega^{2n}M$ is a tn-generated quadratic module and $\Omega^{2n+1}M$ is $(tn+1)$-generated t-fold module, where $\Omega^i M$ is the i-th syzygy of M.

3 The Category $\mathscr{K}_t(\Lambda)$ and $\mathscr{L}_t(\Lambda)$

In this section, we will give some properties of the category of t-Koszul modules. First, by definition it's easy to show that $\mathscr{K}_t(\Lambda)$ and $\mathscr{L}_t(\Lambda)$ are closed under direct sums and direct summands in the category $gr(\Lambda)$.

Proposition 3.1 (i) *Let Λ be an elementary $0,1$-generated algebra and $\{M_i\}_{1\le i\le n}\subseteq gr(\Lambda)$, where $n\ge 1$ is a positive integer. Then $M=\bigoplus_{1\le i\le n}M_i$ is a t-Koszul (resp. t-fold) Λ-module if and only if all M_i's are t-Koszul (resp. t-fold).*

(ii) *Direct product of t-Koszul algebras is also t-Koszul.*

Next we claim that t-fold (resp. t-Koszul) modules are closed under extensions and cokernels of monomorphisms. We omit the proof since it is standard and similar to some proofs of Green, for example, see [5].

Theorem 3.2 *Let Λ be an elementary $0,1$-generated algebra. Then*:

(i) *The category $\mathscr{L}_t(\Lambda)$ is closed under extensions*;

(ii) *The category $\mathscr{L}_t(\Lambda)$ is closed under cokernels of monomorphisms.*

By applying the last theorem, one can show that the category of t-Koszul modules is also closed under extensions and cokernels of monomorphisms. In fact, we have a more general result. Let Λ be a 0,1-generated elementary algebra and $f:N_0\to N_0$ be a strictly increasing function from non-negative integers to non-negative integers. We say that a graded Λ-module X is of *f-type*, if there exists a projective resolution

$$\cdots\to P_n\to\cdots\to P_2\to P_1\to P_0\to X$$

of X in $gr(\Lambda)$, such that P_i is $f(i)$-generated Λ-module for each $i\ge 0$. Note that X is of *f-type* if and only if for any $i\ge 0$, Ω^iX is a $f(i)$-generated $(f(i+1)-f(i)+1)$-fold modules. Now we have:

Theorem 3.3 *Let Λ be a $0,1$-generated elementary algebra, and $0\to X\to Y\to Z\to 0$ be an exact sequence in $gr_0(\Lambda)$. Then*:

(i) *Y is of f-type provided that both X and Z are of f-type*;

(ii) *Z is of f-type provided that both X and Y are of f-type.*

Proof Set $\Omega^0X=X,\Omega^0Y=Y,\Omega^0Z=Z$. Note that in both cases, by applying Theorem 3.2 repeatedly, one gets the following commutative diagrams with exact rows and columns for any non-negative integer n:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{n+1}X & \longrightarrow & P_n & \longrightarrow & \Omega^nX & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{n+1}Y & \longrightarrow & P_n\oplus Q_n & \longrightarrow & \Omega^nY & \longrightarrow & 0,\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{n+1}Z & \longrightarrow & Q_n & \longrightarrow & \Omega^nZ & \longrightarrow & 0.\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here to show that the diagram is exact, we use the standard proof just as in the last theorem. The conclusion follows.

Now considering the function $t(i)$ as defined in Definition 2.1, one gets

Corollary 3.4 *Let Λ be a $0,1$-generated elementary algebra, and $0\to X\to Y\to Z\to 0$ be an exact sequence in $gr_0(\Lambda)$. Then*:

(i) *The category $\mathscr{K}_t(\Lambda)$ is closed under extensions, i.e., if $X,Z\in\mathscr{K}_t(\Lambda)$, then $Y\in\mathscr{K}_t(\Lambda)$*;

(ii) *The category $\mathscr{K}_t(\Lambda)$ is closed under cokernels of monomorphisms, that is to say, if $f:X\to Y$ is a monomorphism with $X,Y\in\mathscr{K}_t(\Lambda)$, then $Y/X\in\mathscr{K}_t(\Lambda)$.*

Remark 3.5 The category of t-Koszul modules may not be closed under kernels of epimorphisms, that is to say, in the exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$, one can not deduce that X is a t-Koszul module provided that Y and Z are both t-Koszul modules. In fact, X may not be a 0-generated module in general. An easy counter-example is the following:

$$\begin{tikzcd} & \overset{1}{\bullet} \arrow[d, "\alpha"'] & \\ \overset{2}{\bullet} \arrow[r, "\beta"] & \overset{3}{\bullet} \arrow[r, "\gamma"] & \overset{4}{\bullet} \end{tikzcd}.$$

Let Q be the above quiver and $\Lambda = kQ/\langle\gamma\beta\rangle$. Set $X = \Lambda e_1/\Lambda\gamma$, $Y = (\Lambda e_1 \oplus \Lambda e_2)/\Lambda(\alpha+\beta)$ and $Z = \Lambda e_2/\Lambda\beta$. It's easy to show that $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is exact. Now Y and Z are both Koszul modules while X is not.

See also [6] Remark 3.4 for counter-examples. Note that in the counter-example there, the module is still Koszul after a grading shift. In general, we have:

Remark 3.6 Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an exact sequence of 0-generated modules with Y and Z both Koszul modules. If, moreover, $X \subseteq \mathrm{Rad}(Y)$, then X is a 1-generated Koszul module.

4 One-point Extensions of t-Koszul Algebras

In this section, we study the Koszulity of the one-point extensions of t-Koszul algebras.

Let Λ be a k-algebra and M a Λ-k-bimodule. The *one-point extension of* Λ *by* M, is defined to be the generalized upper triangular matrix algebra $\left(\begin{smallmatrix} \Lambda & M \\ 0 & k \end{smallmatrix}\right)$, where the addition and multiplication are given by the ones of matrices. Denote this algebra by $\Lambda[M]$.

The category of left $\Lambda[M]$-modules is equivalent to the category $\mathscr{C}(A)$. Recall that an object of $\mathscr{C}(A)$ is a triple $X = (X(\Lambda), V, f)$, where $X(\Lambda)$ is a Λ-module, V is a k-vector space, and $f : M \otimes_k V \to X(\Lambda)$ is a homomorphism of Λ-module. A morphism $\phi : (X(\Lambda), V, f) \to (Y(\Lambda), W, g)$ in $\mathscr{C}(A)$ is a pair (ϕ_1, ϕ_2), where $\phi_1 : X(\Lambda) \to Y(\Lambda)$ is a homomorphism of Λ-module, $\phi_2 : V \to W$ is a k-linear map with the diagram

$$\begin{tikzcd} M \otimes_k V \arrow[r, "f"] \arrow[d, "id\otimes\phi_2"'] & X(\Lambda) \arrow[d, "\phi_1"] \\ M \otimes_k W \arrow[r, "g"'] & Y(\Lambda) \end{tikzcd}$$

commutative.

The equivalence of $\mathscr{C}(A)$ and A-mod is given via the functor $H : \mathscr{C}(A) \to A$-mod defined as follows: For any $X = (X(\Lambda), V, f)$ in $\mathscr{C}(A)$, define an A-module $H(X)$, also identified with X for simplicity, with the underlying vector space $X(\Lambda) \bigoplus V$ and the module action given by

$$\begin{pmatrix} \lambda & m \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} \lambda x + f(m \otimes v) \\ v \end{pmatrix},$$

for any $x \in X(\Lambda), m \in M, v \in V$. Conversely, if we set

$$e_1 = \begin{pmatrix} 1_\Lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then clearly we have $e_2 A e_1 = 0$. Now, for any A-module X, one gets a triple $(X(\Lambda), V, f)$ with $X(\Lambda) = e_1 X$, $V = e_2 X$ and $f : M \otimes_k e_2 X \to e_1 X$ the Λ-modules homomorphism induced by the action of M on the subspace $e_2 X$.

We assume from now on that $\Lambda = \bigoplus_{\geq 0} \Lambda_i$ is an elementary $0, 1$-generated graded k-algebra and $M = \bigoplus_{\geq 1} M_i$ is a graded Λ-module which is generated in degree 1. Then $A = \Lambda[M]$ is an elementary graded k-algebra generated in degree 0 and 1, with gradation $A_0 = \Lambda_0 \oplus k$ and $A_n = \Lambda_n \oplus M_n$ for $n \geq 1$.

Let $\mathscr{C}_0(A)$ denote the full subcategory of $\mathscr{C}(A)$ consisting of the objects $X = (X(\Lambda), V, f)$ such that $X(\Lambda) \in gr(\Lambda)$, V is a finite-dimensional graded vector space, and $f : M \otimes_k V \to X(\Lambda)$ is a degree 0 map. In this case X becomes a graded A-module with the grading inherited from the ones on $X(\Lambda)$ and V.

In fact, we have the following conclusion, it's just the graded version of classical results, more details can be found in [6] and [7].

Proposition 4.1 *$\mathscr{C}_0(A) \overset{H}{\cong} gr(A)$, where $H : \mathscr{C}_0(A) \to gr(A)$ is defined as above.*

Consider the functor $F : gr(\Lambda) \longrightarrow gr(A)$, $X(\Lambda) \mapsto (X(\Lambda), 0, 0)$, where $X(\Lambda) \in gr(\Lambda)$. F is an exact fully faithful functor, and it maps projective Λ-modules to projective A-modules. Moreover, let $X = (X(\Lambda), V, f)$ be an A-module such that $e_2 \cdot X = 0$. Then clearly $V = 0$ and $f = 0$, thus we can regard X as a Λ-module. Via the functor F, the category $gr(\Lambda)$ can be viewed as a full subcategory of $gr(A)$ consisting of the A-modules on which e_2 acts trivially. Let $X(\Lambda) \in gr(\Lambda)$, we also denote $F(X(\Lambda))$ by $X(\Lambda)$ for brevity.

One of our questions is when does $\Lambda[M]$ become a t-Koszul algebra. The following theorem gives an answer. Notice that this is essentially known and is a generalization of Proposition 6.5 in [6].

Theorem 4.2 *Let Λ be an elementary $0,1$-generated graded algebra, and $A = \Lambda[M]$ be the one-point extension of Λ by M, where M is a 1-generated Λ-k-bimodule. Then A is a t-Koszul algebra if and only if Λ is a t-Koszul algebra and $\Omega^1 M[t] \in \mathscr{K}_t(\Lambda)$.*

Proof ($\Rightarrow$) First assume that A is a t-Koszul algebra. Let S_1 and S_2 denote the graded semi-simple A-module associated with the idempotent e_1 and e_2, respectively, where e_1 and e_2 are given as above.

The minimal projective resolution of S_1 in $gr(A)$ coincides with the minimal projective resolution of S_1 in $gr(\Lambda)$, while viewing S_1 as a semi-simple Λ-module. $A = Ae_1 \bigoplus Ae_2$, where $1 = e_1 + e_2$. Now A is a t-Koszul algebra if and only if S_1 and S_2 are both t-Koszul modules. Consider the minimal projective resolution of S_1 in $gr(A)$:

$$\cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S_1 \longrightarrow 0.$$

S_1 is t-Koszul implies that P_i is $t(i)$-generated for any $i \geq 0$. As we pointed out before, such a resolution can also be regarded as a resolution in $gr(\Lambda)$. By definition, it says that Λ is also a t-Koszul algebra.

Next consider the projective resolution of the simple A-module S_2. It's obvious that the projective cover of S_2 in $gr(A)$ is just Ae_2 and the first syzygy $\Omega^1 S_1 = M$. Now M can be viewed as a Λ-module since $e_2 M = 0$. Now let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

be the graded projective resolution of M in $gr(A)$ (and hence in $gr(\Lambda)$). Then

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 = Ae_2 \longrightarrow S_2 \longrightarrow 0$$

gives a minimal projective resolution of S_2 in $gr(A)$.

S_2 being a t-Koszul module means that P_i is generated in degree $t(i)$ for each $i \geq 0$. This is equivalent to $\Omega^1 M[t]$ being a t-Koszul Λ-module.

($\Leftarrow$) Note that the proof of the sufficiency is the same.

5 Modules over One-point Extension Algebras

In this section, Λ is assumed to be an elementary 0,1-generated graded algebra, M a 1-generated Λ-k-bimodule and $A = \Lambda[M]$. We will study the relations between the categories of t-Koszul (t-fold) modules over Λ and A. Also we will study the algebras over which the category of quadratic modules and the category of t-Koszul modules coincide.

First we need the following useful lemma, which was introduced in [6]:

Lemma 5.1 *Let Λ, M, A be as above and $X = (X(\Lambda), V, f)$ be a graded A-module. Then $X \in gr_0(A)$ if and only if $X(\Lambda)_i = 0$ for $i < 0$, $V = V_0$, and $X(\Lambda)$ is generated as a graded* *module by $X(\Lambda)_0$ and the image of $M\otimes V$ under f. In this case, X has a gradation with*

$$X_0 = X(\Lambda_0)\oplus V,\ \text{and } X_i = X(\Lambda)_i \text{ for all } i\geq 0.$$

Let Z denote the submodule of X which is generated by $X(\Lambda)_0$. Clearly Z can be written as $Z=\bigoplus_{\geq 0} Z_i$, where $Z_i=\Lambda_{t(i)}X(\Lambda)_0$ for any $i\geq 0$.

Next if $V=\bigoplus_{i\in Z} V_i$ is a finite-dimensional graded k-vector space, then $(M\otimes V, V, id_{M\otimes V})\in gr(A)$ is a projective module. The degree i component is $(\bigoplus_{j\in Z} M_j\otimes V_{i-j})\oplus V_i$. Let $W = (\mathrm{Im}(f), V, f)$ be the submodule of X which is generated by V. Then there's a projective cover

$$(M\otimes V, V, id_{M\otimes V}) \xrightarrow{(f,\ id)} W$$

in $gr(A)$. Clearly $\mathrm{Ker}(f,\ id)=\mathrm{Ker}(f)$.

The last lemma says that $X\in gr_0(A)$ if and only if $V=V_0$ and $X(\Lambda)=Z+\mathrm{Im}(f)$.

Remark 5.2 (1) This is not a direct sum in general.

(2) In general, $X\in gr_0(A)$ does not imply $X(\Lambda)\in gr_0(\Lambda)$. For example, let $\Lambda=k$ be a graded algebra concentrated in degree 0, and $M=k$ be a graded Λ-module concentrated in degree 1. Then the upper triangular algebra $A=\Lambda[M]=\left(\begin{smallmatrix} k & k\\ 0 & k\end{smallmatrix}\right)$ has a natural grading with $A_0=\left(\begin{smallmatrix} k & 0\\ 0 & k\end{smallmatrix}\right)$ and $A_1=\left(\begin{smallmatrix} 0 & k\\ 0 & 0\end{smallmatrix}\right)$. Consider the projective module $X=\left(\begin{smallmatrix} 0 & k\\ 0 & k\end{smallmatrix}\right)$. Notice that in this case $X(\Lambda)=A_1$ is generated in degree 1 as a Λ-module.

Now assume that $X=(X(\Lambda), V, f)\in gr_0(A)$. Then there is an exact sequence

$$0 \longrightarrow Z \longrightarrow X \xrightarrow{\varphi} X/Z \longrightarrow 0.$$

Clearly,

$$X/Z=(Z+W)/Z\cong W/(W\cap Z)=(\mathrm{Im}(f)/(\mathrm{Im}(f)\cap Z), V, \bar{f}),$$

where $\bar{f}$ is the composition of f and the natural projection $\mathrm{Im}(f)\to \mathrm{Im}(f)/(\mathrm{Im}(f)\cap Z)$. Since $\mathscr{K}_t$ is closed under extensions and closed under cokernels of monomorphisms, it follows that if both Z and X/Z are t-Koszul modules, then X is also a t-Koszul module.

The following proposition was also given in [6]:

Proposition 5.3 *Let $X=(X(\Lambda), V, f)\in gr_0(A)$, Z be the graded Λ-submodule of $X(\Lambda)$ generated by $X(\Lambda)_0$. Then*:

(i) *If $Z\in\mathscr{L}(\Lambda)$ and* $\mathrm{Ker}(M\otimes V\twoheadrightarrow \mathrm{Im}(f)/(\mathrm{Im}(f)\cap Z))$ *is generated in degree* 1, *then $X\in \mathscr{L}(A)$; and conversely, if $X\in\mathscr{L}(A)$ and $Z\in\mathscr{L}(\Lambda)$, then* $\mathrm{Ker}(M\otimes V\twoheadrightarrow \mathrm{Im}(f)/(\mathrm{Im}(f)\cap Z))$ *is* 1*-generated*;

(ii) *If $X(\Lambda)\in\mathscr{L}(\Lambda)$, then $X\in\mathscr{L}(A)$*;

(iii) *If $X(\Lambda)_0=0$, then $X\in\mathscr{L}(A)$ if and only if* $\mathrm{Ker}(\bar{f})$ *is generated in degree* 1;

The next theorem gives the relations between the t-Koszul modules of an algebra and its one-point extensions.

Theorem 5.4 *Let $X=(X(\Lambda), V, f)\in gr_0(A)$, Z be the graded Λ-submodule of $X(\Lambda)$ generated by $X(\Lambda)_0$. Then*:

(i) *Assume that there is an exact sequence*

$$Q_1 \xrightarrow{\psi} M\otimes V \xrightarrow{\bar{f}} \mathrm{Im}f/(\mathrm{Im}f\cap Z) \longrightarrow 0$$

in $gr(\Lambda)$, such that Q_1 is 1-generated projective Λ-module. If $Z\in\mathscr{K}_t(\Lambda)$ and $\mathrm{Ker}(\psi)$ is a t-generated t-Koszul Λ-module, then $X\in\mathscr{K}_t(A)$;

(ii) *Let $\Omega^1(M)$ be the first syzygy of M. If it is a t-generated t-Koszul module, and $X(\Lambda)\in\mathscr{K}_t(\Lambda)$, then $X\in\mathscr{K}_t(A)$*;

(iii) *If $X(\Lambda)_0=0$, then $X\in\mathscr{K}_t(A)$ if and only if $\Omega^1(\mathrm{Ker}(\bar{f}))$ is a t-generated t-Koszul Λ-module.*

Proof (i) Consider the projective cover of $X/Z\cong(\mathrm{Im}(f)/(\mathrm{Im}(f)\cap Z), V, \bar{f})$:

$$(M\otimes V, V, Id_{M\otimes V}) \xrightarrow{(\bar{f},\ id_V)} (\mathrm{Im}(f)/(\mathrm{Im}(f)\cap Z), V, \bar{f}).$$

Clearly $\mathrm{Ker}(\bar{f},\ id_V)=\mathrm{Ker}(\bar{f})$. Now if $\mathrm{Ker}(\psi)$ is a t-generated t-Koszul Λ-module, then it's straightforward to show that X/Z is a t-Koszul module. Thus combining with the assumption that Z is t-Koszul, one shows that X is a t-Koszul module by applying Corollary 3.4.

(ii) Note that the assumption that $\Omega^1(M)$ is a t-generated t-Koszul module implies that A is a t-Koszul algebra. $X(\Lambda) \in \mathscr{K}_t\Lambda$ implies that $X(\Lambda) \in gr_0(\Lambda)$, and hence $Z = X(\Lambda)$. Thus one has $X/Z \cong (0, V, 0)$, which is obviously a t-Koszul A-module since A is a t-Koszul algebra. Now the conclusion follows.

(iii) It's just a special case of (i) if we set $Z = 0$.

Next, we try to answer the question when the category of quadratic modules coincides with the category of t-Koszul modules.

Theorem 5.5 *Let Λ be an elementary $0,1$-generated graded algebra, M a 1-generated Λ-k-bimodule and $A = \Lambda[M]$.*

(i) *Assume that $\mathscr{L}(A) = \mathscr{K}_t(A)$. Then $\mathscr{L}(\Lambda) = \mathscr{K}_t(\Lambda)$, and for every finite-dimensional graded vector space $V = V_0$ and any 1-generated Λ-submodule N of $M \otimes V$, $\Omega^1 N$ is a t-generated quadratic Λ-module*;

(ii) *Assume that $\mathscr{L}(\Lambda) = \mathscr{K}_t(\Lambda)$, and for every finite-dimensional graded vector space $V = V_0$ and any 1-generated Λ-submodule N of $M \otimes V$, $\Omega^1 N$ is a t-generated quadratic Λ-module. If $X = (X(\Lambda), V, f) \in \mathscr{L}(A)$ implies that $Z \in \mathscr{L}(\Lambda)$, where Z is the submodule generated by $X(\Lambda)_0$, then $\mathscr{L}(A) = \mathscr{K}_t(A)$.*

Proof (i) First assume that $\mathscr{L}(A) = \mathscr{K}_t(A)$. It's clear that $\mathscr{L}(\Lambda) = \mathscr{K}_t(\Lambda)$ since the functor F preserves projectively. Now let N be a 1-generated submodule of the module $M \otimes V$. Consider the quotient module

$$X = (M \otimes V, V, id_{M\otimes V})/(N, 0, 0) \cong (M \otimes V/N, V, \pi),$$

where π is the canonical projection. By definition, X is a quadratic module. Now X being t-Koszul implies that $\Omega^1 N = \Omega^2 X$ is a t-generated quadratic A-module, and hence a t-generated quadratic Λ-module.

(ii) Assume that $X = (X(\Lambda), V, f) \in \mathscr{L}(A)$. Then $Z \in \mathscr{L}(\Lambda) = \mathscr{K}_t(\Lambda)$. On the other hand, by Proposition 5.3(i) one gets that $N = \mathrm{Ker}(M \otimes V \twoheadrightarrow \mathrm{Im}(f)/(\mathrm{Im}(f) \cap Z))$ is generated in degree 1, thus by assumption $\Omega^1 N$ is a t-generated quadratic Λ-module and hence a t-generated t-Koszul Λ-module. This implies that X/Z is a t-Koszul A-module, the reason is that $N = \Omega^1(X/Z)$ which has been shown in the proof of Theorem 5.4(i). Now applying Corollary 3.4 one can show that X is a t-Koszul module.

Corollary 5.6 *Let Λ be an elementary $0,1$-generated graded algebra, M a 1-generated Λ-k-bimodule and $A = \Lambda[M]$. Suppose that M satisfies one of the following conditions*:

(i) *$M \subseteq \mathrm{Rad}(P)$ for some 0-generated projective module*;

(ii) *M is projective.*

Then $\mathscr{L}(A) = \mathscr{K}_t(A)$ if and only if $\mathscr{L}(\Lambda) = \mathscr{K}_t(\Lambda)$.

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